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Nonnegative Solutions to Boundary Value Problems for Nonlinear First and Second Order Ordinary Differential Equations

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1. INTRODUCTION

In this paper we use a result by the author [14] to establish sufficient conditions for the existence of nonnegative solutions to first and second order nonlinear ordinary differential equations. We consider the first and second order case with periodic boundary conditions, as well as the Neumann and Picard boundary value problem.

Nonnegative solutions to various boundary value problems for ordinary differential equations, have been considered by Krasnosell'skii [8], Gustafson and Schmitt [5], Schmitt [15], Gatica and Smith [4], Gaines and Santanilla [3], Kolesov [9], Islamov and Shneiberg [6], and Santanilla [14]. Our results, obtained with a unified approach, generalize some of the above.

In Section 2 we state a coincidence degree result to be employed in Sections 3 and 4. This result, which deals with solutions for a semilinear equation of the type $Lx = Nx$ in a convex set, enable us to introduce a unified approach to the boundary value problems mentioned above.

In Section 3 we consider the problem

$$\dot{x}(t) = f(t, x(t)) \tag{2.1}$$

$$x(0) = x(1), \tag{2.2}$$

where f: $[0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $f(0, \cdot) = f(1, \cdot)$. We prove (Theorem 3.2) that if there exist $\alpha > 0$ and $r > 0$ such that $x \cdot f(t, x) \leqslant 0$ for all $x \geqslant 0$ and $\|x\| = r$, and $f(t, x) \geqslant -\alpha x$ for all $x \geqslant 0$ with $\|x\| \leqslant r$, then (2.1), (2.2) has a nonnegative solution. This theorem improves a result in [3] where $\alpha = 1$ and $x \cdot f(t, x) < 0$ for all $x \geqslant 0$ with $\|x\| = r$. We prove a

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new result by reversing the inequality $x \cdot f(t, x) \leqslant 0$ and obtain, as a simple observation, the following result by Mawhin. If there exists $r > 0$ such that for $\|x\| = r$, either $x \cdot f(t, x) \leqslant 0$ or $x \cdot f(t, x) \geqslant 0$, then (2.1), (2.2) has a solution.

In Section 4 we apply our abstract result to the equation $\ddot{x}(t) = f(t, x(t), \dot{x}(t))$ with periodic, Picard and Neumann boundary conditions. We obtain unified results in the sense that the same conditions on f imply the existence of nonnegative solutions to the periodic and Neumann problems. Our main observation here is that the condition imposed on f so that a nonlinear operator associated to f preserves the cone of nonnegative functions, gives bounds for $\dot{x}$ in the case of periodic and Neumann boundary conditions.

2. A Coincidence Theorem in Convex Sets

Let X and Z be real normed linear spaces and L: $\operatorname{dom} L \subset X \to Z$ a linear Fredholm operator of index 0. As a consequence, there exist projections $P: X \to X$ and $Q: Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L$. Further, there exists an isomorphism J: $\operatorname{Im} Q \to \operatorname{Ker} L$, and $L|_{\operatorname{dom} L \cap \operatorname{Ker} P}$ has an inverse which we shall denote by K_p. Let Ω be an open bounded subset of X such that $\operatorname{dom} L \cap \Omega \neq \varnothing$. We assume that N: $\bar{\Omega} \to Z$ is L-compact on $\bar{\Omega}$; i.e., QN and $K_p(I-Q) N$ are compact on $\bar{\Omega}$. It follows [2, 10] that x is a solution to $Lx = Nx$ in $\bar{\Omega}$ if and only if

$$x = Mx \equiv Px + JQNx + K_p(I-Q) Nx.$$

If C is a nonempty closed convex subset of X, then there exists a continuous retraction γ of C. If γ sends bounded sets into bounded sets, we say that C is boundedly retracted. The following theorem gives solutions to $Lx = Nx$ in $C \cap \bar{\Omega}$.

Theorem 2.1 [3]. *Let C be boundedly retracted by γ. If*

(i) $(P + JQN) \gamma(\partial\Omega) \subset C$ *and* $M\gamma(\bar{\Omega}) \subset C$,

(ii) $Lx \neq \lambda Nx$ *for* $x \in (\operatorname{dom} L \cap C \cap \partial\Omega)$ *and* $\lambda \in (0, 1)$, *and*

(iii) $d_B[I - (P + JQN) \gamma|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega, 0] \neq 0$,

then $Lx = Nx$ has a solution in $C \cap \bar{\Omega}$.

Corollary 2.2 [4]. *Let C be boundedly retracted by γ and assume that the follwing conditions are satisfied:*

(i) $\operatorname{Ker} L = \{0\}$;

(ii) $0 \in \Omega \cap C$;

(iii) $Lx \neq \lambda Nx$ *for* $x \in (\operatorname{dom} L \cap C \cap \partial\Omega)$ *and* $\lambda \in (0, 1)$; *and*

(iv) $K_0 N\gamma(\bar{\Omega}) \subset C$.

Then $Lx = Nx$ *has a solution in* $C \cap \Omega$.

This corollary implies the Schauder fixed point theorem when $X = Z$, $C = \bar{\Omega}$ and L is the identity mapping. We shall apply it in the next section to obtain existence of nonnegative periodic solutions of first-order ordinary differential equations.

Our next corollary is an immediate consequence of Corollary 2.2 and will be used in Section 4 to obtain existence of nonnegative solutions to some second order boundary value problems including the periodic case.

COROLLARY 2.3. *Assume that* (i)–(iii) *in Corollary* 2.2 *are satisfied and*

(iv) $K_0 N(C) \subset C$.

Then $Lx = Nx$ *has a solution in* $C \cap \bar{\Omega}$.

3. NONNEGATIVE PERIODIC SOLUTIONS OF FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS

We consider the problem

$$\dot{x}(t) = f(t, x(t)) \tag{2.1}$$

$$x(0) = x(1), \tag{2.2}$$

where f: $[0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $f(0, \cdot) = f(1, \cdot)$.

We seek nonnegative solutions to (2.1), (2.2); i.e., a solution $x = (x_1, x_2, ..., x_n)$ satisfying $x_i(t) \geqslant 0$ on $[0, 1]$. If x is nonnegative we shall write $x(t) \geqslant 0$.

Let

$$X = \{x\colon [0, 1] \to \mathbb{R}^n;\ x \text{ is continuous and } x(0) = x(1)\},$$

$$Z = \{x\colon [0, 1] \to \mathbb{R}^n;\ x \text{ is continuous}\},$$

$$\operatorname{dom} L = \{x \in X\colon \dot{x} \text{ is continuous}\},$$

$$L\colon\ \operatorname{dom} L \to Z, \qquad x \mapsto \dot{x} - \alpha x \quad (\alpha > 0),$$

and

$$N\colon\ X \to Z, \qquad x \mapsto f(\cdot, x(\cdot)) - \alpha x(\cdot).$$

It follows that L is a linear Fredholm operator of index 0 with

$\ker L = \{0\}$ and, using the sup-norm in X and Z, N is L-compact on $\bar{\Omega}$, where $\bar{\Omega}$ is a nonempty open bounded subset of X.

Next we state our first result:

THEOREM 3.1. *Suppose there exist* $\alpha > 0$ *and* $r > 0$ *such that*

(i) $x \cdot f(t, x) \geqslant 0$, *for* $x \geqslant 0$ *and* $\|x\| = r$ *and*

(ii) $f(t, x) \leqslant \alpha x$, *for* $x \geqslant 0$ *and* $\|x\| \leqslant r$.

Then (2.1), (2.2) *has a nonnegative solution* x *such that* $\|x(t)\| \leqslant r$.

Proof. We shall apply Corollary 2.2 with

$$C = \{x \in X: x(t) \geqslant 0 \text{ on } [0.1]\}$$

and

$$\Omega = \{x \in X: \|x(t)\| < r\}.$$

Note that C is boundedly retracted by $\gamma: X \to C$ defined by $(\gamma x)(t) \equiv \gamma_x(t) = (|x_1(t)|, |x_2(t)|, ..., |x_n(t)|)$.

Conditions (i) and (ii) of Corollary 2.2 are trivially satisfied. We check conditions (iii) and (iv). Suppose there exists $x \in \operatorname{dom} L$ with $x(t) \geqslant 0$, $\|x\| = r$, and $\lambda \in (0, 1)$ such that

$$\dot{x}(t) - \alpha x(t) = \lambda(f(t, x(t)) - \alpha x(t)).$$

Then there exists $t_0 \in [0, 1]$ such that $\|x(t_0)\| = r$ and

$$\begin{aligned} 0 &= \frac{d}{dt}(\|x(t)\|^2)\Big|_{t = t_0} \\ &= 2x(t_0) \cdot \dot{x}(t_0) \\ &= 2x(t_0) \cdot [\lambda f(t_0, x(t_0) + (1 - \lambda)\, \alpha x(t_0)] \\ &= 2\alpha(1 - \lambda)\|x(t_0)\|^2 + 2\lambda x(t_0) \cdot f(t_0, x(t_0)) > 0, \end{aligned}$$

and condition (iii) is satisfied.

Finally, it is easy to check that $(K_0 Z)(t) = \int_0^1 G(s, t)\, z(s)\, ds$ for $z \in Z$, where

$$G(s, t) = \frac{1}{e^{-\alpha} - 1} \begin{cases} e^{-\alpha(s - t + 1)}, & 0 \leqslant s < t \\ e^{-\alpha(s - t)}, & t \leqslant s \leqslant 1. \end{cases}$$

Then $(K_0 N \gamma_x)(t) = \int_0^1 G(s, t)[f(s, \gamma_x(s)) - \alpha \gamma_x(s)]\, ds \geqslant 0$ for all $x \in X$ such that $\|x\| \leqslant r$. This completes the proof of the theorem.

The arguments in the proof of Theorem 3.1 suggest

THEOREM 3.2. *Suppose there exist* $\alpha > 0$ *and* $r > 0$ *such that*

(i) $x \cdot f(t, x) \leqslant 0$, *for* $x \geqslant 0$ *and* $\|x\| = r$

(ii) $f(t, x) \geqslant -\alpha x$, *for* $x \geqslant 0$ *and* $\|x\| \leqslant r$.

Then (2.1), (2.2) *has a nonnegative solution* x *such that* $\|x(t)\| \leqslant r$.

This theorem, whose proof is similar to that of Theorem 3.1 with $Lx = \dot{x} + \alpha x$, improves a result in [3, p. 675], where $\alpha = 1$ and $x \cdot f(t, x) < 0$ for all $x \geqslant 0$ with $\|x\| = r$.

Theorem 3.2 is similar to Theorem 7.16 by Kranosel'skii [8], where the Poincaré–Andronov method is used, and hence it is assumed that f has additional properties which guarantee the uniqueness of the solution of the Cauchy problem.

Finally we point out that Corollary 2.2, with $C = X$ and γ being the identity mapping, may be used to obtain a simple proof of the following result in [10, pp. 67–68].

THEOREM 3.3. *Suppose there exists* $r > 0$ *such that either,* $x \cdot f(t, x) \geqslant 0$ *or* $x \cdot f(t, x) \leqslant 0$ *for all* $x \in \mathbb{R}^n$ *with* $\|x\| = r$, *then* (2.1), (2.2) *has at least one solution* (*not necessarily nonnegative*).

4. NONNEGATIVE SOLUTIONS TO BOUNDARY VALUE PROBLEMS FOR SECOND ORDER SYSTEMS

Let $f\colon [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. We shall consider the equation

$$\ddot{x}(t) = f(t, x(t), \dot{x}(t)) \tag{4.1}$$

with one of the boundary conditions

$$x(0) - x(1) = \dot{x}(0) - \dot{x}(1) = 0 \tag{4.2}$$

$$\dot{x}(0) = \dot{x}(1) = 0 \tag{4.3}$$

$$x(0) = x(1) = 0. \tag{4.4}$$

When working with condition (4.2) we assume that $f(0, \cdot, \cdot) = f(1, \cdot, \cdot)$.

Let $G \subset \mathbb{R}^n$ be an open bounded convex set containing the origin. We know that for each $x_0 \in \partial G$ there exists $m(x_0) \in \mathbb{R}^n \setminus \{0\}$ such that $m(x_0) \cdot x_0 > 0$ and

$$\bar{G} \subset \{y\colon m(x_0) \cdot (y - x_0) \leqslant 0\}.$$

Such a vector $m(x_0)$ is called an outer normal to ∂G at x_0.

THEOREM 4.1. *Let G be as above and suppose that*

(i) *for each $x_0 \in \partial G$ with $x_0 \geqslant 0$ there exists an outer normal $m(x_0)$ to ∂G such that if $m(x_0) \cdot y = 0$, $m(x_0) \cdot f(t, x_0, y) \geqslant 0$ and*

(ii) *there exists $\alpha > 0$ such that $f(t, x, y) \leqslant \alpha x$ for all $x \geqslant 0$.*

Then the boundary value problems (4.1), (4.2), *and* (4.1)–(4.3) *have nonnegative solutions.*

Proof. We shall apply Corollary 2.3 and consider Problem (4.1), (4.2). Problem (4.1)–(4.3) is treated similarly. Let

$$X = \{x\colon [0, 1] \to \mathbb{R}^n; x \text{ is } C^1 \text{ and } \dot{x}(0) - \dot{x}(1) = x(0) - x(1) = 0\}$$

with the norm $\|x\| = \max\{\sup\|x(t)\|, \sup\|\dot{x}(t)\|\}$,

$$Z = \{x\colon [0, 1] \to \mathbb{R}^n;\ x \text{ is continuous}\}$$

with the sup-norm,

$$\operatorname{dom} L = \{x \in X\colon x \text{ is of class } C^2\},$$

$$L\colon\ \operatorname{dom} L \to Z, \qquad x \mapsto -\ddot{x} + \alpha x,$$

$$N\colon\ X \to Z, \qquad x \mapsto -f(\cdot, x(\cdot), \dot{x}(\cdot)) + \alpha x(\cdot),$$

and

$$C = \{x \in X\colon x(t) \geqslant 0\}.$$

If we define the ith component, $(\gamma_x)_i(t)$, of $(\gamma_x)(t)$ by

$$(\gamma_x)_i(t) = \begin{cases} x_i(t) & \text{if } \ x_i(t) \geqslant 0 \text{ for all } t \in [0, 1] \\ x_i(t) - \min x_i(t) & \text{if } \ x_i(t) < 0 \text{ for some } t \in [0, 1], \end{cases}$$

then C is boundedly retracted.

It is routine to check that L is Fredholm of index 0, $\operatorname{Ker} L = \{0\}$ and N is L-compact on $\bar{\Omega}$ where Ω is a nonempty open bounded subset of X.

Let $x \in \operatorname{dom} L$ be such that $x(t) \geqslant 0$, $x(t) \in \bar{G}$ and

$$-\ddot{x}(t) + \alpha x(t) = -\lambda f(t, x(t), \dot{x}(t)) + \lambda \alpha x(t) \tag{4.5}$$

for some $\lambda \in (0, 1)$. Then, by (ii),

$$\ddot{x}(t) \leqslant \alpha x(t). \tag{4.6}$$

Let $\tilde{\alpha}_i$ be such that

$$\alpha \int_0^1 x_i(s)\, ds \leqslant \tilde{\alpha}_i$$

and choose $\tau \in (0, 1)$ so that $\dot{x}_i(\tau) = 0$. We have

$$\dot{x}_i(0) = \dot{x}_i(1) = \int_\tau^1 \ddot{x}_i(s)\, ds \leqslant \tilde{\alpha}_i,$$

and

$$-\dot{x}(1) = -\dot{x}_i(0) = \int_0^\tau \ddot{x}_i(s)\, ds \leqslant \tilde{\alpha}_i.$$

Hence,

$$-\dot{x}_i(t) \leqslant \alpha \int_t^1 x_i(s)\, ds - \dot{x}_i(1) \leqslant 2\tilde{\alpha}_i.$$

Similarly, (4.6) implies that $\dot{x}_i(t) \leqslant 2\tilde{\alpha}_i$. Therefore, there exists $\tilde{\alpha}$ such that $\|\dot{x}(t)\| < \tilde{\alpha}$.

Let

$$\Omega = \{x \in X\colon x(t) \in G \text{ and } \|\dot{x}(t)\| < \tilde{\alpha} + 1\}$$

and suppose that (4.5) is satisfied for some $x \in \operatorname{dom} L \cap C \cap \partial\Omega$. Then there exists $t_0 \in [0, 1]$ such that $x_0 \equiv x(t_0) \in \partial G$. Let $g(t) = m(x_0) \cdot [x(t) - x_0]$. Since $g(t) \leqslant 0 = g(t_0)$, it follows that $0 = g'(t_0) = m(x_0) \cdot \dot{x}(t_0)$, and

$$\begin{aligned} 0 \geqslant g''(t_0) &= m(x_0) \cdot \ddot{x}(t_0) \\ &= m(x_0) \cdot [\lambda f(t_0, x(t_0), \dot{x}(t_0)) + (1-\lambda)\, \alpha x(t_0)] \\ &= \lambda m(x_0) \cdot f(t_0, x(t_0), \dot{x}(t_0)) + (1-\lambda)\, \alpha m(x_0) \cdot x_0 \\ &> 0. \end{aligned}$$

Hence, condition (iii) of Corollary 2.3 is satisfied.

Finally, it is easily shown that

$$(K_0 z)(t) = \int_0^1 G(s, t)\, z(s)\, ds,$$

where

$$G(s, t) = \frac{1}{2\sqrt{\alpha}(e^{\sqrt{\alpha}} - 1)} \begin{cases} e^{\sqrt{\alpha}(t-s)} + e^{\sqrt{\alpha}(1+s-t)} & 0 \leqslant s \leqslant t \\ e^{\sqrt{\alpha}(s-t)} + e^{\sqrt{\alpha}(1+t-s)} & t < s \leqslant 1. \end{cases}$$

Thus,

$$(K_0 Nx)(t) = \int_0^1 G(s, t)[\alpha x(s) - f(s, x(s), \dot{x}(s))]\, ds \geqslant 0$$

for all $x \in C$, and (iv) in Corollary 2.3 is satisfied. The theorem follows from Corollary 2.3.

The arguments in the proof of Theorem 4.1 are essentially valid in the case of Neumann boundary conditions. Here we know that the Green's function associated to $-\ddot{x} + \alpha x = z$, $\alpha > 0$, is given by

$$G(s, t) = \frac{1}{\sqrt{\alpha} \sinh \sqrt{\alpha}} \begin{cases} \cosh \sqrt{\alpha}\, s \cosh \sqrt{\alpha}(1-t), & 0 \leqslant s \leqslant t \\ \cosh \sqrt{\alpha}\, t \cosh \sqrt{\alpha}\,(1-s), & t < s < 1. \end{cases}$$

The last inequality in (i) of Theorem 4.1 may be reversed by using an inner normal vector, and the inequality in (ii) may be reversed as in the next theorem.

THEOREM 4.2. *Let $G \subset \mathbb{R}^n$ be an open bounded convex set containing the origin and suppose that*

(i) *for each $x_0 \in \partial G$ with $x_0 \geqslant 0$ there exists an inner normal $m(x_0)$ to ∂G such that if $m(x_0) \cdot y = 0$, $m(x_0) \cdot f(t, x_0, y) < 0$ and*

(ii) *there exists α, $0 < \alpha \leqslant 8$ such that $f(t, x, y) \geqslant -\alpha x$ for all $x \geqslant 0$.*

Then (4.1), (4.2) *has nonnegative solution.*

Proof. This result follows from Theorem 2.1 and the proof is a combination of arguments already introduced, hence we shall omit the details. Using the notation in Theorem 4.1 with $Lx = \ddot{x}$ and $Nx = f(\cdot, x(\cdot), \dot{x}(\cdot))$, it follows that condition (ii) in Theorem 2.1 is satisfied as in Theorem 4.1. Conditions (iii) and (i) are satisfied as in [3, Theorem 4.1]. Q.E.D.

Remarks. (1) Theorem 4.2 is an extension of Theorem 4.5 in [3], where f is independent of $\dot{x}$.

(2) A growth condition on $\|f\|$ and assumption (i) in Theorem 4.2 were introduced by Bebernes and Schmitt [1] to obtain solutions (not necessarily nonnegatives) to (4.1), (4.2). (See also Gaines and Mawhin [2], for further references.)

(3) The technique of reducing the original equation to one with trivial kernel has been considered, using a different approach, by Keller [7] and Pennline [13].

(4) Corollary 2.3 with $C = X$ and γ being the identity mapping, may be used to obtain Corollary V.5 in [12], for solutions which are not necessarily nonnegative.

(5) To study Eq. (4.1) with boundary condition (4.4) we need, in addition to (i) and (ii) of Theorem 4.1, an assumption giving bounds for $\dot{x}$. Specifically, we have

THEOREM 4.2. *Let us assume that the conditions* (i) *and* (ii) *of Theorem* 4.1 *hold and moreover that*

(iii) *there exist* $\alpha_1, \alpha_2 \geqslant 0$, $\beta \in L^1([0, 1]; \mathbb{R})$ *with* $\beta(t) > 0$, *such that*

$$f(t, x, y) \geqslant -\alpha_1 x - \alpha_2 y - \beta(t) \tag{4.7}$$

for all $y \geqslant 0$, $x \geqslant 0$, *and* $x \in \bar{G}$.

Then Problem (4.1)–(4.4) *has a nonnegative solution.*

Proof. We apply Corollary 2.3 and proceed as in the proof of Theorem 4.1 with $X = \{x\colon [0, 1] \to \mathbb{R}^n;\ x \text{ is } C^1 \text{ and } x(0) = x(1) = 0\}$. Let $x \in \operatorname{dom} L$ be such that $x(t) \geqslant 0$, $x(t) \in \bar{G}$ and

$$-\ddot{x}(t) + \alpha x(t) = -\lambda f(t, x(t), \dot{x}(t)) + \lambda \alpha x(t) \tag{4.8}$$

for some $\lambda \in (0, 1)$.

Let $\tau \in (0, 1)$ be such that $\dot{x}(\tau) = 0$. Combining (4.7) and (4.8) we have

$$-\ddot{x}(t) \leqslant \lambda(\alpha_1 x(t) + \alpha_2 \dot{x}(t) + \beta(t)),$$

and integrating over $[0, \tau]$ (resp. $[\tau, 1]$), we obtain $\dot{x}_i(0) \leqslant \beta_i$ (resp. $-\dot{x}_i(1) \leqslant \beta_i$) for some $\beta_i > 0$. Thus, integrating (4.6) over $[0, t]$ (resp. $[t, 1]$) we have $\dot{x}_i(t) \leqslant \tilde{\alpha}_i + \beta_i$ (resp. $-\dot{x}_i(t) \leqslant \tilde{\alpha}_i + \beta_i$). We conclude that there exists $\tilde{\alpha}$ such that $\|\dot{x}(t)\| \leqslant \tilde{\alpha}$. For the reader's convenience we mention that

$$G(s, t) = \frac{1}{\sqrt{\alpha} \sinh \sqrt{\alpha}} \begin{cases} \sinh \sqrt{\alpha}\,(1 - t) \sinh \sqrt{\alpha}\, s, & 0 \leqslant s \leqslant t \\ \sinh \sqrt{\alpha}\,(1 - s) \sinh \sqrt{\alpha}\, t, & t < s \leqslant 1, \end{cases}$$

and so $G(s, t) \geqslant 0$. Q.E.D.

In order to obtain a second result for the Picard problem, we shall use a condition different from (i) in Theorem 4.2. We consider

$$\ddot{x}(t) + f(t, x(t), \dot{x}(t)) = 0 \tag{4.9}$$

$$x(0) = x(1) = 0, \tag{4.4}$$

where $f\colon [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. It has been proved in [14, Theorem 4.6] that Problem (4.9)–(4.4), with f independent of $\dot{x}$, has a nonnegative solution if the following conditions are satisfied.

(i) There exist $\alpha \in (0, 1)$ and $g \in L^1([0, 1]; \mathbb{R}_+)$ such that $x \cdot f(t, x) \leqslant \alpha \|x\|^2 + g(t)\|x\|$ for all $x \geqslant 0$;

(ii) $f(t, x) \geqslant -\alpha x$ for all $x \geqslant 0$ with $\|x\| \leqslant R$ where $R > (1 - \alpha)^{-1} \|g\|_1$.

Note that condition (i) implies that

$$\overline{\lim_{\substack{\|x\| \to \infty \\ x \geqslant 0}}} \frac{x \cdot f(t, x)}{\|x\|^2} < 1.$$

In the next theorem this limit is $\leqslant \Pi^2$, the first eigenvalue of $\ddot{x} + \alpha x = 0$, $x(0) = x(1) = 0$.

THEOREM 4.3. *Assume that the following conditions are satisfied*:

(i)

$$F(t) = \overline{\lim_{\substack{\|x\| \to \infty \\ x \geqslant 0}}} \left(\sup_{y \in \mathbb{R}^n} \frac{x \cdot f(t, x, y)}{\|x\|^2} \right) \leqslant \Pi^2$$

uniformly in $t \in [0, 1]$.

(ii) $\int_0^1 F(t) \sin^2 \Pi t \, dt < \Pi^2/2$.

(iii) *There exists* $\alpha < \Pi^2$ *such that* $f(t, x, y) \geqslant \alpha x$ *for all* $x \geqslant 0$.

(iv) *For each* $r > 0$, *there exists* $\alpha_1, \alpha_2 \geqslant 0$, $\beta \in L^1([0, 1]; \mathbb{R})$ *with* $\beta(t) > 0$ *such that*

$$f(t, x, y) \leqslant \alpha_1 x + \alpha_2 y + \beta(t)$$

for all $y \geqslant 0$ *and* $x \geqslant 0$ *with* $\|x\| \leqslant r$.

Then Problem (4.9)–(4.4) *has a nonnegative solution.*

Proof. Once again we apply Corollary 2.3 and proceed as in Theorem 4.2 with $Lx = -\ddot{x} - \alpha x$ and $(Nx)(t) = f(t, x(t), \dot{x}(t)) - \alpha x(t)$.

Assumptions (i) and (ii) imply, as in [11, Theorem 2], that there exists $r > 0$ such that if $x \in \operatorname{dom} L$, $x(t) \geqslant 0$, $\lambda \in (0, 1)$ and $Lx = \lambda Nx$, then $\|x(t)\| < r$. Further, by (iii) and (iv), there exists $\tilde{\alpha}$ such that $\|\dot{x}(t)\| \leqslant \tilde{\alpha}$. Hence, we define

$$\Omega = \{x \in X : \|x(t)\| < r, \|\dot{x}(t)\| < \tilde{\alpha} + 1\}.$$

Here we have that the Green's function $G_\alpha(s, t)$ associated to $-\ddot{x} - \alpha x = z$ with $x(0) = x(1) = 0$, is given by

$$G_\alpha(s, t) = \frac{1}{\sqrt{\alpha} \sin \sqrt{\alpha}} \begin{cases} \sin \sqrt{\alpha}\, s \sin \sqrt{\alpha}\, (1 - t), & 0 \leqslant s \leqslant t \\ \sin \sqrt{\alpha}\, t \sin \sqrt{\alpha}\, (1 - s), & t < s \leqslant 1, \end{cases}$$

if $0 < \alpha < \Pi^2$,

$$G_0(s, t) = \begin{cases} s(1 - t), & 0 \leqslant s \leqslant t \\ t(1 - s), & t < s \leqslant 1 \end{cases}$$

and

$$G_\alpha(s,t)=\frac{1}{\sqrt{-\alpha}\sinh\sqrt{-\alpha}}\begin{cases}\sinh\sqrt{-\alpha}\,(1-t)\sinh\sqrt{-\alpha}\,s, & 0\leqslant s\leqslant t\\ \sinh\sqrt{-\alpha}\,(1-s)\sinh\sqrt{-\alpha}\,t, & t<s\leqslant 1,\end{cases}$$

if $\alpha<0$. Thus $G_\alpha(s,t)\geqslant 0$ and the result follows from Corollary 2.3. Q.E.D.

Conditions (i) and (ii) in Theorem 4.3 are those in [11, Theorem 2] for solutions which are not necessarily nonnegative. It follows from Theorem 2 [11] that the elementary problem

$$\ddot{x}+\beta x=\sin \Pi t \tag{4.10}$$

$$x(0)=x(1)=0 \tag{4.4}$$

has a solution if $\beta<\Pi^2$. However, multiplying (4.10) by $\sin \Pi t$ and integrating over [0, 1], ones check that (4.10)–(4.4) has no nonnegative solution, showing that a condition of the type (iii) in Theorem 4.3 cannot be omitted.

A combination of the arguments in the proof of Theorems 4.1 and 4.3 gives

THEOREM 4.4. *Assume that conditions* (i), (ii), *and* (iii), *with* $\alpha<0$, *in Theorem* 4.3, *are satisfied. Then Problems* (4.9)–(4.2) *and* (4.9)–(4.3) *have nonnegative solutions.*

We point out that if f satisfies $f(t,x,y)\leqslant \alpha x$, for some $\alpha<0$, in Eq. (4.9), then none of the above problems have positive solutions. In fact, if $x\in C^2$,

$$(\dot{x}(t)\cdot x(t))'=\ddot{x}(t)\cdot x(t)+\|\dot{x}(t)\|^2.$$

Integrating and using the boundary conditions we obtain a contradiction if $x(t)$ is a solution to(4.9) such that $x(t)\geqslant 0$ and $x\not\equiv 0$.

We close this section by observing that in our results for nonnegative solution we do not require $f\geqslant 0$ as in [4, 5, 8, 15].

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